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Exact solution and surface critical behaviour of open cyclic SOS lattice models

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Abstract. We consider the L-state cyclic solid-on-solid lattice models under a class of open boundary conditions. The integrable boundary face weights are obtained by solving the reflection equations. Functional relations for the fused transfer matrices are presented for both periodic and open boundary conditions. The eigenspectra of the unfused transfer matrix is obtained from the functional relations using the analytic Bethe ansatz. For a special case of crossing parameter $\lambda = \pi/L$, the finite-size corrections to the eigenspectra of the critical models are obtained, from which the corresponding conformal dimensions follow. The calculation of the surface free energy away from criticality yields two surface specific heat exponents, $\alpha_s = 2 - L/2\ell$ and $\alpha_1 = 1 - L/\ell$, where $\ell = 1, 2, \ldots, L-1$ coprime to L. These results are in agreement with the scaling relations $\alpha_s = \alpha_b + \nu$ and $\alpha_1 = \alpha_b - 1$.

1. Introduction

Square lattice models in statistical mechanics with non-periodic boundary conditions have received intermittent attention over the years (see, e.g., [1–6]). Up until quite recently the systematic study of the integrability of such non-periodic systems lagged well behind the study of the corresponding periodic systems. It is well understood that models with periodic boundary conditions are integrable when their bulk/Boltzmann weights satisfy the Yang–Baxter equation [7]. Since Sklyanin's work [8], we now understand that lattice models with open boundary conditions are integrable if in addition the boundary weights satisfy the reflection equations [9]. In particular, Sklyanin formulated the construction of commuting transfer matrices for the six-vertex model with open boundary conditions, from which the integrability is assured. Recent lectures on subsequent developments can be found in [10, 11].

Beyond their intrinsic mathematical interest, exactly solvable lattice models with open boundary conditions are attractive from the viewpoint of studying various surface critical phenomena [12–21]. Our motivation here is to study the surface critical behaviour of square lattice cyclic solid-on-solid (CSOS) models [22, 23]. These models are face models in which the adjacency condition between neighbouring heights is defined by the Dynkin diagram of the affine $A^{(1)}_{L-1}$ algebra. The CSOS model has been well studied for periodic boundary conditions. The free energy, the local height probabilities and the correlation length have all been evaluated, along with their corresponding bulk critical exponents [22–25]. The complete operator content has been discussed in [26] and the fusion procedure has been carried out in [27, 28]. Some surface properties have been derived in [29].

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The crossing or anisotropy parameter of the CSOS models is defined by $\lambda = \ell\pi/L$, where $\ell = 1, 2, \ldots, L-1$ is coprime to L [23]. A special case of interest is $L = 3$ and $\ell = 2$ which is related to Baxter's three colourings of the square lattice [7, 23, 30, 31].

The layout of this paper is as follows. In section 2 the CSOS models with both periodic and open boundary conditions are described. We solve the reflection equations for the boundary face weights. The functional relations of the fused transfer matrices are also presented. In section 3 the eigenspectra of the unfused transfer matrix is extracted from the functional relations following the analytic Bethe ansatz method. The finite-size corrections to the transfer matrix eigenspectra at criticality are obtained for a special value of the crossing parameter. In section 4 the free energy of the open boundary models is shown to satisfy a unitarity relation. We solve the unitarity relation following the inversion relation method [7, 32]. From the singular part of the free energy we obtain two surface specific heat exponents in agreement with scaling predictions. We conclude with a brief discussion.

2. CSOS models

The CSOS lattice models [22, 23] are a family of L-state face models [7] built on the affine $A^{(1)}_{L-1}$ Dynkin diagram. States at adjacent sites of the square lattice must be adjacent on the Dynkin diagram. The cyclic nature of the heights distinguishes the CSOS model from the corresponding RSOS model [33] built on the A_L Dynkin diagram.

2.1. Bulk face weights

The allowed, or non-zero, face weights of the CSOS models are given by [22, 23]

$$\begin{aligned}
W\left(\begin{matrix} a\mp 1 & a \\ a & a\pm 1\end{matrix}\middle| u\right) &= \frac{\vartheta_1(\lambda-u)}{\vartheta_1(\lambda)} \\
W\left(\begin{matrix} a & a\mp 1 \\ a\pm 1 & a\end{matrix}\middle| u\right) &= \left[\frac{\vartheta_4(w_{a-1})\vartheta_4(w_{a+1})}{\vartheta_4^2(w_a)}\right]^{1/2}\frac{\vartheta_1(u)}{\vartheta_1(\lambda)} \\
W\left(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a\end{matrix}\middle| u\right) &= \frac{\vartheta_4(w_a\pm u)}{\vartheta_4(w_a)}
\end{aligned} \tag{2.1}$$

where $w_a = a\lambda + w_0$. The height $a = 1, 2, \ldots, L$ and $0 < w_0 < \pi$ is a free parameter. The crossing parameter λ is given by $\lambda = \ell\pi/L$, where $\ell = 1, 2, \ldots, L-1$ is coprime to L and $L > 2$. The elliptic functions $\vartheta_1(u)$, $\vartheta_4(u)$ are standard theta functions of nome p

$$\vartheta_1(u) = \vartheta_1(u, p) = 2p^{1/4}\sin u\prod_{n=1}^{\infty}(1-2p^{2n}\cos 2u + p^{4n})(1-p^{2n}) \tag{2.2}$$

$$\vartheta_4(u) = \vartheta_4(u, p) = \prod_{n=1}^{\infty}(1-2p^{2n-1}\cos 2u + p^{4n-2})(1-p^{2n}) \tag{2.3}$$

where $0 < p < 1$ with $p = 0$ at criticality.

These face weights satisfy the star-triangle equation

$$\begin{aligned}
&\sum_g W\left(\begin{matrix} f & g \\ a & b\end{matrix}\middle| u\right) W\left(\begin{matrix} e & d \\ f & g\end{matrix}\middle| v\right) W\left(\begin{matrix} d & c \\ g & b\end{matrix}\middle| v-u\right) \\
&\qquad = \sum_g W\left(\begin{matrix} e & g \\ f & b\end{matrix}\middle| v-u\right) W\left(\begin{matrix} g & d \\ a & b\end{matrix}\middle| v\right) W\left(\begin{matrix} e & d \\ g & c\end{matrix}\middle| u\right)
\end{aligned} \tag{2.4}$$

inversion/unitarity relations

$$\sum_g W\left(\begin{matrix} d & g \\ a & b \end{matrix}\middle| u\right) W\left(\begin{matrix} d & c \\ g & b \end{matrix}\middle| -u\right) = \rho(u)\delta_{a,c} \tag{2.5}$$

and the crossing unitarity relations

$$\sum_g W\left(\begin{matrix} g & b \\ d & a \end{matrix}\middle| \lambda - u\right) W\left(\begin{matrix} c & b \\ d & g \end{matrix}\middle| \lambda + u\right) \frac{\vartheta_4(w_a)\vartheta_4(w_g)}{\vartheta_4(w_d)\vartheta_4(w_b)} = \rho(u)\delta_{a,c} \tag{2.6}$$

where $\rho(u) = \vartheta_1(\lambda - u)\vartheta_1(\lambda + u)/\vartheta_1^2(\lambda)$.

2.2. Periodic boundaries

There is a hierarchy of commuting families of transfer matrices constructed by the fusion procedure on the CSOS models under periodic boundary conditions. Let

$$W_{m\times n}\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right)$$

be the fused face weights with fusion levels m and n in the vertical and horizontal directions, respectively [27, 28]. Then the fused transfer matrices $\boldsymbol{T}^{(m,n)}(u)$ are defined with elements

$$\langle \boldsymbol{a}|\boldsymbol{T}^{(m,n)}(u)|\boldsymbol{b}\rangle = \prod_{j=1}^{N} W_{m\times n}\left(\begin{matrix} a_j & a_{j+1} \\ b_j & b_{j+1} \end{matrix}\middle| u\right) \tag{2.7}$$

with $a_{N+1} = a_1$ and $b_{N+1} = b_1$ where N is the number of faces in a row of the lattice. By construction the fused face weights satisfy the star-triangle equations, resulting in the commutation relations

$$[\boldsymbol{T}^{(m,n)}(u), \boldsymbol{T}^{(m,\bar{n})}(v)] = 0. \tag{2.8}$$

These fused transfer matrices satisfy groups of functional relations, which can be easily proved by fusion. Let us define

$$\begin{aligned} &\boldsymbol{T}_k^{(n)} = \boldsymbol{T}_{(m,n)}(u + k\lambda) \\ &\boldsymbol{T}^{(n)} = 0 \qquad \text{if } n < 0 \text{ or } m < 0 \\ &\boldsymbol{T}^{(0)} = \mathbf{I} \end{aligned} \tag{2.9}$$

along with the function

$$f_n^m = \prod_{j=0}^{m-1} \rho^N(u - j\lambda + n\lambda). \tag{2.10}$$

Then for each $m = 1, 2, \ldots$ the functional relations are

$$\boldsymbol{T}_0^{(n)}\boldsymbol{T}_n^{(1)} = \boldsymbol{T}_0^{(n+1)} + f_{n-1}^m \boldsymbol{T}_0^{(n-1)} \qquad n = 1, 2, \ldots. \tag{2.11}$$

The unfused models of interest here are recovered by setting the fusion level to $n = m = 1$. Disregarding finite-size corrections, the bulk free energy in this case satisfies

$$T(u)T(u + \lambda) = f_0^1. \tag{2.12}$$

This is the unitarity relation for periodic boundary conditions.

2.3. Boundary face weights

Integrable models with open boundary conditions are defined by both the bulk and the boundary face weights. The latter are represented by three heights interacting around a triangular face [11, 34–37]. For the CSOS models,

$$K\left(a\,{}^{c}_{b}\middle|\right)=0 \qquad \text{unless } |a-b|=1, L-1 \text{ and } |a-c|=1, L-1 \tag{2.13}$$

which satisfy the boundary version of the star-triangle equation (reflection equations)

$$\begin{aligned}\sum_{f,g} W\left(\begin{matrix}a & b\\ g & c\end{matrix}\middle|u-v\right) K\left(g\,{}^{c}_{f}\middle|u;\xi\right) W\left(\begin{matrix}a & g\\ d & f\end{matrix}\middle|u+v\right) K\left(d\,{}^{f}_{e}\middle|v;\xi\right)\\ =\sum_{f,g} K\left(b\,{}^{c}_{f}\middle|v;\xi\right) W\left(\begin{matrix}a & b\\ g & f\end{matrix}\middle|u+v\right) K\left(g\,{}^{f}_{e}\middle|u;\xi\right) W\left(\begin{matrix}a & g\\ d & e\end{matrix}\middle|u-v\right).\end{aligned} \tag{2.14}$$

In general there may be more arbitary parameters than ξ. Inserting the CSOS bulk face weights (2.2) into the reflection equations and making use of the elliptic function identity

$$\begin{aligned}\vartheta_1(x+y)\vartheta_1(x-y)\vartheta_4(w+v)\vartheta_4(w-v)-\vartheta_1(v+y)\vartheta_1(v-y)\vartheta_4(w+x)\vartheta_4(w-x)\\ =\vartheta_1(x+v)\vartheta_1(x-v)\vartheta_4(w+y)\vartheta_4(w-y)\end{aligned} \tag{2.15}$$

we find the following CSOS boundary face weights:

$$K\left(1\,{}^{L}_{L}\middle|u;\xi\right)=\frac{\vartheta_1[\xi+u]\vartheta_4[u-(w_L+\xi)]}{\vartheta_1^2(\lambda)} \tag{2.16}$$

$$K\left(1\,{}^{1}_{1}\middle|u;\xi\right)=\frac{\vartheta_1[\xi-u]\vartheta_4[u+(w_1+\xi)]}{\vartheta_1^2(\lambda)} \tag{2.17}$$

$$K\left(a\,{}^{t}_{b}\middle|u;\xi\right)=\frac{\vartheta_1[\xi+(a-t)u]\vartheta_4[u-(a-t)(w_b+\xi)]}{\vartheta_1^2(\lambda)}\delta_{b,t}. \tag{2.18}$$

The identity (2.15) also plays a role in establishing the integrability of the bulk weights [23].

It is obvious that the boundary face weights satisfy the crossing symmetry

$$\sum_c\sqrt{\frac{\vartheta_4(w_c)}{\vartheta_4(w_a)}}W\left(\begin{matrix}d & c\\ a & b\end{matrix}\middle|2u+\lambda\right)K\left(c\,{}^{e}_{b}\middle|u+\lambda\right)=\frac{\vartheta_1(2u+2\lambda)}{\vartheta_1(\lambda)}K\left(a\,{}^{e}_{b}\middle|-u\right). \tag{2.19}$$

2.4. Fusion results

The fused transfer matrices $\boldsymbol{T}^{(m,n)}(u)$ of the open boundary CSOS models are defined by the following elements,

$$\begin{aligned}\langle\boldsymbol{a}|\boldsymbol{T}^{(m,n)}(u)|\boldsymbol{b}\rangle=\sum_{\{c_0,\ldots,c_N\}}K_+^{(n)}\left(\begin{matrix}a_0\\ b_0\end{matrix}c_0\middle|u\right)K_-^{(n)}\left(c_N\,{}^{a_N}_{b_N}\middle|u\right)\\ \times\prod_{k=0}^{N-1}\left[W_{(n\times m)}\left(\begin{matrix}c_k & c_{k+1}\\ b_k & b_{k+1}\end{matrix}\middle|u\right)W_{(m\times n)}\left(\begin{matrix}c_k & a_k\\ c_{k+1} & a_{k+1}\end{matrix}\middle|u+n\lambda-\lambda\right)\right]\end{aligned} \tag{2.20}$$

where the right boundary fused face weights $K_-^{(n)}$ are given by fusing

$$K_-\left(a\,{}^{t}_{b}\middle|u\right)=K\left(a\,{}^{t}_{b}\middle|u;\xi_-\right) \tag{2.21}$$

while the left boundary fused face weights $K_+^{(n)}$ are given by fusing

$$K_+\left(\begin{smallmatrix} t & \\ b & a\end{smallmatrix}\middle| u\right) = K\left(a \begin{smallmatrix} t \\ b\end{smallmatrix}\middle| -u+\lambda; \xi_+\right)\sqrt{\frac{\vartheta_4^2(w_a)}{\vartheta_4(w_t)\vartheta_4(w_b)}}. \tag{2.22}$$

These fused transfer matrices form commuting families,

$$[\boldsymbol{T}^{(m,n)}(u), \boldsymbol{T}^{(m,n')}(v] = 0. \tag{2.23}$$

Like its periodic counterpart, the unfused transfer matrix is recovered by setting the fusion levels to $m = n = 1$, i.e. $\boldsymbol{T}(u) = \boldsymbol{T}^{(1,1)}(u)$.

The fusion procedure for face models with open boundary conditions has been demonstrated elsewhere [35, 37] and thus we do not repeat the details here. However, it is worthwhile writing down the functional relations satisfied by the fused CSOS transfer matrices. We find

$$\begin{aligned} &\boldsymbol{T}_k^{(m,n)} = \boldsymbol{T}_{(m,n)}(u+k\lambda) \\ &\boldsymbol{T}^{(m,n)} = 0 \qquad \text{if } n < 0 \text{ or } m < 0 && (2.24)\\ &\boldsymbol{T}^{(m,0)} = \mathbf{I} && (2.25)\end{aligned}$$

where now the auxiliary function f_n^m is determined by

$$f_n^m = \frac{\omega^-(u+n\lambda)\omega^+(u+n\lambda)}{\rho(2u+2n\lambda)}\prod_{j=0}^{m-1}\rho^N(u-j\lambda+n\lambda)\rho^N(u+j\lambda+n\lambda). \tag{2.26}$$

The boundaries contribute the diagonal matrix factors $\omega^-(u)$ and $\omega^+(u)$, with

$$\begin{aligned}\omega^-_{c_r,c_r}(u) = \sum_{a,b}&\sqrt{\frac{\vartheta_4(w_b)}{\vartheta_4(w_{c_r-1})}}\\ &\times W\left(\begin{matrix} c_r & b \\ c_r-1 & a\end{matrix}\middle| 2u+\lambda\right) K_-\left(b \begin{smallmatrix} c_r \\ a\end{smallmatrix}\middle| u+\lambda\right) K_-\left(c_r-1 \begin{smallmatrix} a \\ c_r\end{smallmatrix}\middle| u\right)\end{aligned} \tag{2.27}$$

$$\begin{aligned}\omega^+_{c_l,c_l}(u) = \sum_{a,d}&\sqrt{\frac{\vartheta_4^2(w_a)\vartheta_4^2(w_{c_l})}{\vartheta_4^3(w_{c_l-1})\vartheta_4(w_d)}}\\ &\times W\left(\begin{matrix} c_l & c_l-1 \\ d & a\end{matrix}\middle| \lambda-2u\right) K_+\left(\begin{smallmatrix} c_l & \\ a & \end{smallmatrix} c_l-1\middle| u+\lambda\right) K_+\left(\begin{smallmatrix} a & \\ c_l & \end{smallmatrix} d\middle| u\right).\end{aligned} \tag{2.28}$$

Here height c_r (c_l) is located on the right (left) boundary. The matrix functions $\omega^\pm(u)$ are simplified under the crossing symmetry (2.19), with

$$\omega^\mp_{c,c}(u) = \frac{\vartheta_1(2\lambda \pm 2u)}{\vartheta_1(\lambda)}\sum_a K\left(c-1 \begin{smallmatrix} c \\ a\end{smallmatrix}\middle| \mp u; \xi_\mp\right) K\left(c-1 \begin{smallmatrix} a \\ c\end{smallmatrix}\middle| \mp u; \xi_\mp\right). \tag{2.29}$$

Then the functional relations have similar forms to the periodic boundary case, namely

$$\boldsymbol{T}_0^{(m,n)}\boldsymbol{T}_n^{(m,1)} = \boldsymbol{T}_0^{(m,n+1)} + f_{n-1}^m \boldsymbol{T}_0^{(m,n-1)} \qquad m, n \geqslant 0. \tag{2.30}$$

In this way, again after dropping the finite-size corrections, we arrive at the unitarity relation

$$T(u)T(u+\lambda) = f(u) \tag{2.31}$$

where the function f is given by $f(u) = f_0^1(u)\rho(2u)\vartheta_1^2(\lambda)/\vartheta_1^2(2\lambda)$ after appropriate renormalization of the free energies. This relation is sufficient to determine both the bulk and surface free energies [37].

3. Exact Bethe ansatz solution

Sklyanin presented the algebraic Bethe ansatz solution of the six-vertex model [8] or spin-$\frac{1}{2}$ *XXZ* chain [5] with open boundary conditions. Unfortunately, the generalization of the algebraic Bethe ansatz to treat other integrable open boundary models has not made much progress, in particular, for models in which the arrow or spin reversal symmetry is broken. However, when such symmetry holds, the Bethe ansatz solutions of many integrable open boundary models have been obtained (see, e.g., [10, 14, 38]). Here we show that the analytic ansatz method [39] can be applied to find the transfer matrix eigenspectra of the CSOS models. This method has also been applied with success to the CSOS models with periodic boundary conditions [24].

3.1. Bethe ansatz solution

Consider the transfer matrix $\boldsymbol{T}\left(\begin{matrix} a_0 & a_N \\ b_0 & b_N \end{matrix}\middle| u\right)$, which is the transfer matrix $\boldsymbol{T}_0^{(1,1)}$ with fixed heights b_N, a_N along the right boundary and b_0, a_0 along the left boundary. For the solutions (2.16)–(2.18) the transfer matrix is non-zero only for $b_0 = a_0$ and $b_N = a_N$. Suppose $\boldsymbol{T}_{b,d}(u) = \boldsymbol{T}\left(\begin{matrix} b & d \\ b & d \end{matrix}\middle| u-\lambda\right)$. Let us consider the following ansatz,

$$\begin{aligned} T_{b,d}(u) = K\left(d-1\,\begin{matrix} d \\ d \end{matrix}\middle| u;\xi_-\right) K\left(b+1\,\begin{matrix} b \\ b \end{matrix}\middle| u;\xi_+\right) \frac{\vartheta_1(2u-2\lambda)}{\vartheta_1(2u-\lambda)} \\ \times\vartheta_1^{2N}(\lambda-u)\prod_{j=1}^{M}\frac{\vartheta_1(u+\lambda+u_j)\vartheta_1(u-u_j)}{\vartheta_1(u-\lambda-u_j)\vartheta_1(u+u_j)} \\ \times K\left(d+1\,\begin{matrix} d \\ d \end{matrix}\middle| u-\lambda;\xi_-\right) K\left(b-1\,\begin{matrix} b \\ b \end{matrix}\middle| u-\lambda;\xi_+\right) \frac{\vartheta_1(2u)}{\vartheta_1(2u-\lambda)} \\ \times\vartheta_1^{2N}(u)\prod_{j=1}^{M}\frac{\vartheta_1(u-\lambda+u_j)\vartheta_1(u-2\lambda-u_j)}{\vartheta_1(u-\lambda-u_j)\vartheta_1(u+u_j)} \end{aligned} \tag{3.1}$$

for the eigenspectra of the transfer matrix $\boldsymbol{T}_{b,d}(u)$. We can check that the above ansatz satisfies the functional relation (2.30) if the parameters u_j satisfy

$$T_{b,d}(u_k) = 0 \qquad k = 1, 2, \ldots, M. \tag{3.2}$$

As a result, the ansatz (3.1) should give the eigenspectra of the CSOS transfer matrix, with (3.2) as the related Bethe ansatz equations. The eigenspectra of the fused transfer matrices can be written down according to the fusion results from (3.1) and (3.2).

3.2. Finite-size corrections

At criticality the CSOS models are equivalent to the six-vertex model. The transfer matrix $\boldsymbol{T}_{b,d}(u)$ is then independent of the boundary heights (as are the fused transfer matrices) and the Bethe ansatz solution (3.1)–(3.2) is of the same form as the solution of the open boundary six-vertex model [8, 14, 16]. Now the finite-size corrections to the fused transfer matrices of the $U_q(sl_2)$-invariant six-vertex model have been derived in [19]. Thus by a similar calculation we obtain the finite-size corrections to the fused transfer matrices of the critical CSOS models.

For simplicity we consider the limit $\xi_\pm \to \mathrm{i}\infty$ with $\lambda = \pi/L$ and N even. The fused CSOS transfer matrix eigenvalues then behave like

$$\log T^{(p,p)}(u) = -2N f_\mathrm{b}(u) - f_\mathrm{s}(u) + \frac{\pi}{12N}(c - 24\Delta_{1,\nu,r})\sin(Lu) + o\left(\frac{1}{N}\right). \tag{3.3}$$

The central charge is given by

$$c = \frac{3p}{p+2} - \frac{6p}{L(L-p)} \tag{3.4}$$

where $p = 1, 2, \ldots, L-2$ labels the fusion level. The conformal weights are given by

$$\Delta_{1,\nu,r} = \frac{(L-(L-p)r)^2 - p^2}{4Lp(L-p)} + \frac{\nu(p-\nu)}{2p(p+2)} \tag{3.5}$$

with ν a unique integer determined by

$$\nu = r - 1 - p\left\lfloor \frac{r-1}{p} \right\rfloor. \tag{3.6}$$

and $r = 1, 3, \ldots \leqslant L-2$. Here $\lfloor x \rfloor$ is the largest integer part less than or equal to x. The functions $f_\mathrm{b}(u)$ and $f_\mathrm{s}(u)$ are the bulk free energy and surface free energy of the critical models, respectively, which are also calculated for the off-critical models in the next section. They are not given explicitly here.

4. Surface free energy and critical exponents

The bulk and surface free energies of the CSOS models can be found from the unitarity relation (2.31) with certain analyticity assumptions, as has been shown in the study of the eight-vertex model [7, 20, 32].

The unitarity relation (2.31) combines the inversion relation and crossing symmetries of the local bulk and boundary face weights. We can separate the contributions from the bulk and surface free energies in this relation [37]. Let $T(u) = T_\mathrm{b}(u)T_\mathrm{s}(u)$ be the eigenvalues of the transfer matrix $\boldsymbol{T}(u)$. Define $T_\mathrm{b} = \kappa_\mathrm{b}^{2N}$ and $T_\mathrm{s} = \kappa_\mathrm{s}$, then the free energies are defined by $f_\mathrm{b}(u) = -\log\kappa_\mathrm{b}(u)$ and $f_\mathrm{s}(u) = -\log\kappa_\mathrm{s}(u)$. We have

$$\kappa_\mathrm{b}(u)\kappa_\mathrm{b}(u+\lambda) = \frac{\vartheta_1(\lambda-u)\vartheta_1(\lambda+u)}{\vartheta_1(\lambda)\vartheta_1(\lambda)} \tag{4.1}$$

for the bulk and

$$\begin{aligned}\kappa_\mathrm{s}(u)\kappa_\mathrm{s}(u+\lambda) = {} & \frac{\vartheta_1(2\lambda+2u)\vartheta_1(2\lambda-2u)}{\vartheta_1^2(2\lambda)} \\ & \times K\left(d-1\,\begin{matrix}d\\d\end{matrix}\middle|-u;\xi_-\right) K\left(d-1\,\begin{matrix}d\\d\end{matrix}\middle|u;\xi_-\right) \\ & \times K\left(b-1\,\begin{matrix}b\\b\end{matrix}\middle|-u;\xi_+\right) K\left(b-1\,\begin{matrix}b\\b\end{matrix}\middle|u;\xi_+\right)\end{aligned} \tag{4.2}$$

for the surface. Here height d (b) is located on the right (left) boundary.

To solve the unitarity relations it is convenient to introduce the new variables

$$\begin{array}{lll} x = \mathrm{e}^{-\pi\lambda/\epsilon} & w = \mathrm{e}^{-2\pi u/\epsilon} & q = \mathrm{e}^{-\pi^2/\epsilon} \\ v_a = \mathrm{e}^{-\pi w_a/\epsilon} & v_\pm = \mathrm{e}^{-\pi\xi_\pm/\epsilon} & p = \mathrm{e}^{-\epsilon} \end{array} \tag{4.3}$$

along with the conjugate modulus transformation of the theta functions,

$$\vartheta_1(u, \mathrm{e}^{-\epsilon}) = \rho(u, \epsilon) E(\mathrm{e}^{-2\pi u/\epsilon}, \mathrm{e}^{-2\pi^2/\epsilon}) \tag{4.4}$$

$$\vartheta_4(u, \mathrm{e}^{-\epsilon}) = \rho(u, \epsilon) E(-\mathrm{e}^{-2\pi u/\epsilon}, \mathrm{e}^{-2\pi^2/\epsilon}). \tag{4.5}$$

The factor $\rho(u, \epsilon)$ is harmless and will be disregarded, while

$$E(z, x) = \prod_{n=1}^{\infty} (1 - x^{n-1} z)(1 - x^n z^{-1})(1 - x^n). \tag{4.6}$$

Suppose that $\kappa_{\mathrm{b}}(w)$ is analytic and non-zero in the annulus $x^2 \leqslant w \leqslant 1$, we can Laurent expand $f_{\mathrm{b}}(w)$ as $\log \kappa_{\mathrm{b}}(w) = \sum_{n=-\infty}^{\infty} c_n w^n$. Then inserting the series expansion into the logarithm of both sides of (4.1) and equating coefficients in powers of w gives [23]

$$f_{\mathrm{b}}(w, p) = -\sum_{n=1}^{\infty} \frac{(x^{2n} + q^{2n} x^{-2n})(1 - w^n)(1 - x^{2n} w^{-n})}{n(1 + x^{2n})(1 - q^{2n})}. \tag{4.7}$$

Similarly, taking the Laurent expansion $\log \kappa_{\mathrm{s}}(w) = \sum_{n=-\infty}^{\infty} c_n w^n$ and solving the unitarity relation (4.2) yields

$$\begin{aligned} f_{\mathrm{s}}(w, \xi_\pm, p) = {} & \sum_{n=1}^{\infty} \frac{(-1)^n (v_+^{2n} v_b^{2n} + q^{2n} v_+^{-2n} v_b^{-2n} + v_-^{2n} v_d^{2n} + q^{2n} v_-^{-2n} v_d^{-2n})(w^n + x^{2n} w^{-n})}{n(1 + x^{2n})(1 - q^{2n})} \\ & + \sum_{n=1}^{\infty} \frac{(v_+^{2n} + q^{2n} v_+^{-2n} + v_-^{2n} + q^{2n} v_-^{-2n})(w^n + x^{2n} w^{-n})}{n(1 + x^{2n})(1 - q^{2n})} \\ & - \sum_{n=}^{\infty} \frac{(x^{4n} + q^{2n} x^{-4n})(1 - w^{2n})(1 - x^{4n} w^{-2n})}{n(1 + x^{4n})(1 - q^{2n})} \\ & - \sum_{n=1}^{\infty} \frac{4(x^{2n} + q^{2n} x^{-2n})}{n(1 - q^{2n})}. \end{aligned} \tag{4.8}$$

The surface free energy is explicitly dependent on the boundary heights b, d and $\xi_\pm$.

The specific heat critical exponents may be obtained from the leading order singularity of the free energies. In practice, the singular behaviour is extracted by means of the Poisson summation formula [7]. For the bulk free energy it follows that [22, 23]

$$f_{\mathrm{b}}(w, p) \sim p^{\pi/\lambda} \qquad \text{as } p \to 0. \tag{4.9}$$

When $\ell = 1$ and L is even there is a multiplicative $\log p$ factor. It follows that the bulk specific heat exponent of the CSOS models is $\alpha_{\mathrm{b}} = 2 - \pi/\lambda$ where we recall that $\lambda = \ell\pi/L$. The same idea can be applied to find the surface specific heat exponents from the surface energy $f_{\mathrm{s}}(w, \xi_\pm, p)$. Following [40, 41] and the treatment of the eight-vertex model [20] we define the excess internal energy e_{s},

$$e_{\mathrm{s}}(p) \sim \frac{\partial f_{\mathrm{s}}(w, \xi_\pm, p)}{\partial p} + e_1(p) \tag{4.10}$$

where $e_1(p)$ is called the local internal energy in the surface layer, which is the correction energy to the surface internal energy $e_{\mathrm{s}}(p)$, given by

$$e_1(p) \sim \frac{\partial f_{\mathrm{s}}(w, \xi_\pm, p)}{\partial \xi_\pm}. \tag{4.11}$$

The free parameter $\xi_\pm$ appearing in the boundary face weights can be interpreted as a surface coupling. This has been explicitly shown in the study of the eight-vertex model [20]. The corresponding specific heats are defined by

$$C_s \sim \frac{\partial e_s}{\partial p} \qquad C_1 \sim \frac{\partial e_1}{\partial p}. \tag{4.12}$$

Application of the Poisson summation formula to $f_s(w, \xi_\pm, p)$ yields

$$e_s(p) \sim p^{\pi/2\lambda - 1} \tag{4.13}$$
$$e_1(p) \sim p^{\pi/\lambda} \tag{4.14}$$

as $p \to 0$, with a similar $\log p$ correction factor as for the bulk case. The surface specific heat exponents of the CSOS models follow as

$$\alpha_s = 2 - \frac{\pi}{2\lambda} \qquad \text{and} \qquad \alpha_1 = 1 - \frac{\pi}{\lambda}. \tag{4.15}$$

For the three colouring problem ($L = 3$ and $\ell = 2$) we thus obtain the values $\alpha_s = \frac{5}{4}$ and $\alpha_1 = -\frac{1}{2}$.

Recalling the bulk exponent $\alpha_b = 2 - \pi/\lambda$ [22, 23] and the correlation length exponent $\nu = \pi/2\lambda$ [24] we are thus able to provide a further significant test of the scaling relations [40, 41]

$$\alpha_s = \alpha_b + \nu \qquad \text{and} \qquad \alpha_1 = \alpha_b - 1. \tag{4.16}$$

5. Discussion

In this paper we have derived exact results for the critical surface properties of the CSOS lattice models. They can be generalized to the fused CSOS models. In this and related work [20, 21, 37] the crossing unitarity relation plays a key role in deriving the surface free energy away from criticality. The CSOS lattice models with fixed boundary conditions on the square lattice with diagonal orientation can be treated in a similar manner by incorporating inhomogeneities into the bulk face weights and taking appropriate values of the inhomogeneities and the boundary couplings $\xi_\pm$, as has been explicitly demonstrated for the ABF RSOS models [21]. However, we do not pursue this direction here as the change in lattice orientation does not affect the critical exponents.

The excess surface critical exponent α_s has been obtained from the singular leading term of the excess internal energy e_s. It turns out that the singular leading term does not depend on the boundary face weights and thus α_s is independent of the details of the boundary weights. This behaviour has already been seen in the study of the ABF model [21].

Other models of immediate interest are the dilute A_L models [42, 43] which can be obtained from Kuniba's $A_2^{(2)}$ face model [44] under appropriate restriction. The boundary face weights have been found for these models and the surface critical properties can thus be studied in a similar way [45].

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